EXISTENCE OF BASIS IN SOME WHITNEY SPACES

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By

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ABSTRACT

EXISTENCE OF BASIS IN SOME WHITNEY SPACES

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Existence of basis in locally convex spaces has been a hot subject in functional analysis for more than 40 years. We will give some partial solutions to this well-known problem. We will demonstrate two cases of Cantor-type sets with extension property under some special cases. These sets are K_N with (DN) where $\limsup \alpha_n < N$ and K_∞ with (DN) where $\limsup \alpha_n < \infty$.

Keywords: Schauder basis, nuclear Fréchet spaces, power series space of infinite type, Whitney spaces, Cantor-type sets, topological invariants (DN), (Ω), β and diametral dimension.

ÖZET

TABANI OLAN BAZI WHITNEY UZAYLARI

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Lokal konveks uzaylarının tabanı olup olmadığı konusu, fonksiyonel analizde kırk yılı aşkın bir süredir popüler bir konu olagelmiştir. Bu tezde tabanı olan bazı Whitney uzaylarına örnekler verilecektir. Bu örnekler genişletme özelliği olan bir kısım Cantor tipi kümelerdir. Bu kümeler $\limsup \alpha_n < N$ olan K_N ve $\limsup \alpha_n < \infty$ olan K_∞'dir.

Anahtar sözcükler: Schauder tabanı, nükleer Fréchet uzayları, Whitney uzayları, Cantor tipi kümeler, topolojik değişmez (DN), (Ω), β ve çapsal boyut .

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Contents

1 Introduction **1**

2 Preliminaries **3**

2.1 Locally Convex Spaces 3

2.2 Spaces of Whitney Functions 10

2.3 Some Topological Invariant Properties 10

2.3.1 (DN) : 11

2.3.2 (Ω) : 11

2.3.3 Diametral Dimension 12

2.3.4 Topological Invariant $\boldsymbol{\beta}$ 13

2.4 Cantor-Type Sets, Generalized Cantor Sets 17

3 Existence of Basis in Some Whitney Spaces **23**

Chapter 1

Introduction

Existence of basis problem in locally convex spaces has been a hot subject in functional analysis for more than 40 years. It was shown by Zobin and Mityagin [10] that there are nuclear Fréchet spaces without a basis. Since their counter example was a constructed one, it is still not known whether there exists a concrete example of nuclear function space without a basis. Many researchers believe that if there exists one, it will be found among Whitney spaces. We will give some partial solutions to this well-known problem. We will work on topological vector spaces, and especially locally convex spaces which are rich in theory. Among locally convex spaces, we will focus mainly on nuclear Fréchet spaces. Some basic definitions and theorems will be given. Then Whitney spaces and generalized Cantor-type sets will be defined. We will also define some topological invariants which are (DN), (Ω), diametral dimension and β and some theorems related to isomorphism of locally convex spaces which will be of greatest importance in obtaining solutions to existence of basis problem. The relation between β and diametral dimension and asymptotical behavior of β for Cantor-type sets will be useful in obtaining solutions to existence problem.

By the help of a theorem due to A. Aytuna, J. Krone and Terzioğlu [2] we will show two examples of Whitney spaces with a basis. These Whitney spaces are defined on Cantor-type sets.

For both of the cases, first we will give a proof with basic ideas and techniques. Then a detailed proof will be given. Although the Cantor-type sets K_N and K_∞ are not like each other topologically, while showing that both $\varepsilon(K_N)$ and $\varepsilon(K_\infty)$ have basis we use same proof with minor changes.

We will show only existence of basis for both Whitney spaces, a concrete example of a basis will not be given. In [5] Goncharov constructs a basis in the spaces of Whitney functions $\varepsilon(K)$ for two model cases, where $K \subset \mathbb{R}$ is a sequence of closed intervals tending to a point.

Chapter 2

Preliminaries

2.1 Locally Convex Spaces

In topological vector spaces, we consider generally locally convex spaces which are rich in theory. Among locally convex spaces, we will focus mainly on nuclear Fréchet spaces. In this section some basic definitions and theorems will be given.

Definition 2.1.1 *Let E be a topological vector space, $\mathcal{U}_0$ be the set of all neighborhoods of the origin. Then a subset $\mathcal{V}$ of $\mathcal{U}_0$ is called **a basis of neighborhoods of the origin** if for any $U \in \mathcal{U}_0$, there exists $V \in \mathcal{V}$ such that $V \subset U$. A topological vector space which has a basis of convex neighborhoods of the origin is called a **locally convex topological vector space**, or in short form **locally convex space**.*

Definition 2.1.2 *Let E be a locally convex space, and $\mathcal{U}$ be a collection of zero neighborhoods in E. If for any zero neighborhood U in E, there exists a $V \in \mathcal{U}$ and $\epsilon > 0$ such that $\epsilon.V \subset U$, then $\mathcal{U}$ is called a **Fundamental System of Zero Neighborhoods**.*

Definition 2.1.3 *Let E be a locally convex space and $(||.||_\alpha)_{\alpha \in A}$ be a family of continuous seminorms in E.*

*If the sets $\{U_\alpha := x \in E : ||x||_\alpha < 1, \alpha \in A\}$ form a fundamental system of zero neighborhoods in E, then the family of seminorms $(||.||_\alpha)_{\alpha \in A}$ is called a **fundamental system of seminorms** in E.*

If E is a vector space, and a family $\mathcal{V} = (V_\alpha)_{\alpha \in I}$ of convex neighborhoods of the origin is given, then there is a coarsest topology on E compatible with the algebraic structure in which every set in $\mathcal{V}$ is a neighborhood. Under this topology E becomes a locally convex space and the sets;

$$V_{I',\epsilon} = \epsilon. \bigcap_{i \in I'} V_i \qquad (I' \text{: a finite set in } I, V_i \in \mathcal{V})$$

form a basis of neighborhoods of the origin. (For proof, see [12])

Similarly, if on a vector space E a family $\mathcal{S} = (p_i)_{i \in I}$ of seminorms is given, then there is a coarsest topology on E compatible with the algebraic structure in which every seminorm in $\mathcal{S}$ is continuous under the topology generated by the following sets (which form a fundamental system of zero neighborhoods);

$$\{x : \sup_{i \in I'} p_i(x) \leq \epsilon \ \ (\epsilon > 0, I' : \text{ a finite set in } I)\}$$

and under this topology, E becomes a locally convex space. (For proof, see [12]).

If a Hausdorff locally convex space E has a countable fundamental system of seminorms (or countable fundamental system of zero neighborhoods), then topology of E can be defined by a translation invariant metric, i.e. E is metrisable. This metric d can be defined as follows:
Let $(||.||_n)_{n \in \mathbb{N}}$ be the countable fundamental system of seminorms (if we are given $(U_n)_{n \in \mathbb{N}}$ as the countable fundamental system of zero neighborhoods, then consider the corresponding Minkowski functionals $(||.||_n)_{n \in \mathbb{N}}$ as follows:
$||x||_n = ||x||_{U_n} := \inf\{t > 0 : x \in t.U_n\}$ Clearly, these $||x||_n$'s are seminorms).
Then:

Theorem 2.1.4 *Let E be a Hausdorff locally convex space. Then the following are equivalent:*

1. *E has a countable fundamental system of zero neighborhoods.*

2. *E has a countable fundamental system of seminorms.*

3. *E is metrisable.*

Proof: Equivalence of (1) and (2) follows from the definitions.
(3) $\Rightarrow$ (1) : Define $U_n = \{x \in E : d(x,0) < \frac{1}{n},\ n \in \mathbb{N}\}$. Then $(U_n)_{n\in\mathbb{N}}$ form a countable fundamental system of zero neighborhoods.
(2) $\Rightarrow$ (3) : Let $(||.||_n)_{n\in N}$ be a countable fundamental system of seminorms. Then define $d : E \times E \to \mathbb{R}$ as follows:

$$d : (x,y) \mapsto \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||x-y||_n}{1+||x-y||_n}\ , \quad x,y \in E.$$

It is shown in [12] that d is a metric. Since $d(x+a, y+a) = d(x,y)$ for any $a \in E$, d is invariant under translation. Let us show that topology generated by d, τ_d is equivalent to topology $\tau_{||.||_n}$ defined by countable fundamental system of seminorms. To show this; let $O_n = \{x \in E : d(x,0) < \frac{1}{n}, n \in \mathbb{N}\}$ be the usual basis of neighborhoods in E with respect to τ_d. Then we will construct an open set O'_n in E (open with respect to $\tau_{||.||_n}$) such that $O'_n \subset O_n$ which will give us that $\tau_{||.||_n}$ is finer than τ_d. O'_n is constructed as follows:
Let O_n be given. Since $\sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$, fix N such that $\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{1}{2\cdot n}$. Then define $O'_{n,i} = \{x \in E : \frac{||x||_i}{1+||x||_i} < \frac{1}{2\cdot n}\}$ where $1 \le i \le N$ and let $O'_n = \bigcap_{1\le i\le N} O'_{n,i}$. First, note that O'_n is not empty since $0 \in x \in O'_{n,i}$ for all $1 \le i \le N$ and finite intersection of open sets is again open. So O'_n is an open set containing 0. Now let us show that $O'_n \subset O_n$:
Take any $x \in O'_n$, then

$$\begin{aligned} d(x,0) = & \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{||x||_k}{1+||x||_k} \\ = & \sum_{k=1}^{N} \frac{1}{2^k} \frac{||x||_k}{1+||x||_k} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \underbrace{\frac{||x||_k}{1+||x||_k}}_{<1} < \end{aligned}$$

$$\sum_{k=1}^{N} \frac{1}{2^k} \frac{||x||_k}{1+||x||_k} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} < \left(\sum_{k=1}^{N} \frac{1}{2^k} \cdot \frac{1}{2 \cdot n} \right) + \frac{1}{2 \cdot n} < \frac{1}{n} \left(\sum_{k=1}^{N} \frac{1}{2^{k+1}} + \frac{1}{2} \right) < \frac{1}{n} \cdot 1 < \frac{1}{n} \quad . \tag{2.1}$$

This means that $x \in O_n$ which implies that $O'_n \subset O_n$. So $\tau_{||.||_n}$ is finer than τ_d.

To show converse;

Let $U_n = \{x \in E : ||x||_n < 1\}$ be the basis of neighborhoods of E with respect to $\tau_{||.||_n}$. Now we will construct an open set U'_n in E (open with respect to τ_d) such that $U'_n \subset U_n$ which will give us that τ_d is finer than $\tau_{||.||_n}$. U'_n is constructed as follows:

Let U_n be given, then let $U'_n = \{x \in E : d(x,0) < \frac{1}{2^{n+1}}\}$. Clearly U'_n is an open set with respect to τ_d. Take any $x \in U'_n$, then

$$\frac{1}{2^n} \cdot \frac{||x||_n}{1+||x||_n} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{||x||_k}{1+||x||_k} = d(x,0) < \frac{1}{2^{n+1}}$$

which implies that

$$\frac{1}{2^n} \cdot \frac{||x||_n}{1+||x||_n} < \frac{1}{2^{n+1}} \Rightarrow ||x||_n < 1$$

which means that x is in U_n, i.e. $U'_n \subset U_n$. So τ_d is finer than $\tau_{||.||_n}$. Therefore we get that τ_d and $\tau_{||.||_n}$ define the same topology.

□

Definition 2.1.5 *A locally convex space E which is metrisable and complete is called a* ***Fréchet Space****.*

If E is a Fréchet Space, then by (2.1.4), E is metrisable. Since E is metrisable, it is sufficient to work with sequences. If $(\|.\|_k)_{k=1}^{\infty}$ is a fundamental system of seminorms, E is complete if and only if for every sequence $\{x_n\}$ in E such that $\lim_{n,m\to\infty} \|x_n - x_m\|_k = 0$ for every k, there is an element $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\|_k = 0$ for every k.

Definition 2.1.6 *A locally convex space E is said to be* ***nuclear****, if for each absolutely convex zero neighborhood U in E there exist an absolutely convex zero neighborhood V and a measure μ on the σ^*-compact set V^0, so that*

$$\|x\|_U \leq \int_{V^0} |y(x)| d\mu(y) \text{ for all } x \in E.$$

Definition 2.1.7 *A sequence $(e_n)_{n\in\mathbb{N}}$ in a locally convex space E is called a* ***basis*** *if for every $e \in E$, there exists a unique sequence $(a_n)_{n\in\mathbb{N}}$ ($a_n \in \mathbb{R}$) such that the series $\sum_{n=0}^{\infty} a_n e_n$ converges to e (here by convergence we mean convergence in topology of E, i.e. for any neighborhood U of e there exists $N \in \mathbb{N}$ such that for all $N' \geq N$, $\left(\sum_{n=0}^{N'} a_n e_n\right)$ is in $U-$neighborhood of e).*
If $(e_n)_{n\in\mathbb{N}}$ is a basis of E, we can define ***coefficient functionals*** *of E as follows:*

$$u_n(x) = a_n, \text{ for all } x \text{ in } E \text{ where } x = \textstyle\sum_{n=0}^{\infty} a_n e_n.$$

Then a basis $(e_n)_{n\in\mathbb{N}}$ is called a ***Schauder basis*** *if its coefficient functionals u_n are continuous.*
Let $(e_n)_{n\in\mathbb{N}}$ be a Schauder basis of E. If for each continuous seminorm p on E there is a continuous seminorm q on E and a $C > 0$ such that

$$\sum_{n\in\mathbb{N}} |u_n(x)| p(e_n) \leq Cq(x) \text{ for all } x \in E ,$$

then $(e_n)_{n\in\mathbb{N}}$ is called an ***absolute basis****.*

Definition 2.1.8 *Two locally convex spaces E and F are called* ***isomorphic*** *if there exists a one-to-one linear mapping from E onto F which is continuous in both directions.*

Theorem 2.1.9 *{Dynin - Mityagin Theorem} If E is a nuclear Fréchet space then every Schauder basis $(e_n)_{n\in\mathbb{N}}$ is absolute, i.e., for each fundamental system of seminorms $(\|\ \|_k)_{k\in\mathbb{N}}$ we have: for every $k \in \mathbb{N}$ there is an $l \in \mathbb{N}$ and a $C > 0$ with*

$$\sum_{n=1}^{\infty} |a_n(x)| . ||e_n||_k \leq C||x||_l \text{ for all } x \in E.$$

For proof of last theorem, we will refer to [8].

Definition 2.1.10 *Let A be a set consisting of sequences $a = (a_n)_{n\in\mathbb{N}}$ of positive numbers such that the following conditions hold:*

1. *For any $m \geq 1$, there exists a sequence $a \in A$ such that $a_m > 0$*
2. *For any finite set of elements $a^{(k)} \in A, \quad k = 1, 2, \ldots, m$, there exists a sequence $a = (a_n)_{n\in\mathbb{N}} \in A$ such that*

$$\max\{a_n^{(k)} \ : \ 1 \leq k \leq m\} \ \leq a_n \text{ for all } n \geq 1.$$

*Then the set A is called a **Köthe set**.*

For any Köthe set A, define

$$\lambda(A) = \{x = (x_n)_{n\in\mathbb{N}} \ : \ p_a(x) = \sum_{n=1}^{\infty} a_n|x_n| < \infty \text{ for all } a = (a_n)_{n\in\mathbb{N}} \in A\}. \quad (2.2)$$

*The set $\lambda(A)$ is called the **Köthe space determined by** A or in short form **Köthe space**.*

It's a well known fact that all Köthe spaces have a basis $(e_n)_{n\in\mathbb{N}}$ defined as follows:

$$e_n = (\delta_{n,i})_i \text{ where } \delta_{n,i} = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

By the topology determined by the family of seminorms $(p_a)_{a\in A}$ defined in (2.2), $\lambda(A)$ becomes a locally convex space. In case A is a countable set, $\lambda(A)$ becomes a Fréchet space.

Definition 2.1.11 *Let A be a Köthe set satisfying following conditions:*

1. *For any $a = (a_n) \in A$,*

$$0 < a_n \leq a_{n+1} \textit{ for all } n \geq 1,$$

2. *For every* $a = (a_n) \in A$ *there is a* $b = (b_n) \in A$ *such that*

$$a_n^2 \leq b_n \text{ for all } n \geq 1.$$

Then A is called a ***power set of infinite type*** *and the Köthe space determined by A is called a $G_\infty-$space.*
Let $\alpha = (\alpha_n)_{n=0}^\infty$ be an increasing sequence of positive numbers, then we can define a power set of infinite type A by the help of α as follows;

$$A = \{\beta_k = (\beta_{k,n})_n \ : \beta_{k,n} = k^{\alpha_n} \ k = 1, 2, \ldots\}$$

Then the $G_\infty-$space determined by A is called a ***power series space of infinite type*** *and denoted by $\Lambda_\infty(\alpha)$.*

Note that the space $\Lambda_\infty(\alpha)$ is a Fréchet space. We can rewrite $\Lambda_\infty(\alpha)$ for a given increasing sequence $\alpha = (\alpha_n)$ of positive numbers as follows:

$$\Lambda_\infty(\alpha) = \left\{ x = (x_n) : \quad \sum_{n=1}^{\infty} k^{\alpha_n} \cdot |x_n| < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

Definition 2.1.12 *The space of rapidly decreasing sequences s is defined as follows:*

$$s = \{x = (x_n) : ||x||_q = \sum_{n=1}^{\infty} |x_n| n^q < \infty \ \forall \ q\} \tag{2.3}$$

The space s becomes a Fréchet space under the topology defined by the sequence of seminorms given in (2.3). s plays a prominent role in the theory of nuclear Fréchet spaces. In fact s is a power series of infinite type determined by the sequence $(\log n)_{n=1}^\infty$, i.e.

$$s = \Lambda_\infty\left((\log n)_n\right). \tag{2.4}$$

Equivalence of (2.3) and (2.4) is proved in [19].

2.2 Spaces of Whitney Functions

Definition 2.2.1 *Let $K \subset \mathbb{R}$ be a compact set. The space of functions $f : K \to \mathbb{R}$ extendable to C^∞-functions on $\mathbb{R}$ equipped with the topology defined by the sequence of norms*

$$||f||_q = |f|_q + \sup\{|\left(R_y^q f\right)^{(i)}(x)| \cdot |x-y|^{i-q} : x, y \in K, x \neq y,\ i \leq q\}, q = 0, 1 \ldots \quad (2.5)$$

where $|f|_q = \sup\{|f^{(j)}(x)| : x \in K, j \leq q\}$ *and*

$$R_y^q f(x) = f(x) - T_y^q f(x) = f(x) - \sum_{k=0}^{q} \frac{f^{(k)}(y)}{k!}(x-y)^k \text{ (the Taylor remainder)}$$

is called ***space of infinitely differentiable Whitney functions on*** *K and denoted by $\varepsilon(K)$.*

2.3 Some Topological Invariant Properties

Now let us give definitions of some topological invariant properties in locally convex spaces; (DN), (Ω), diametral dimension, and β. Diametral dimension was one of the first topological invariants introduced by C. Bessaga, A. Pelczyński and S. Rolewicz [3]. (DN) was introduced by Vogt [17] (see also the class D_1 in [22]), topological invariant (DN) coincides with the condition (d_3) derived by E. Dubinsky [4] for Köthe spaces; equivalence was shown by Vogt[17]. (Ω) was first used by Vogt and Wagner [18] and β was founded by Zahariuta in [20] by generalizing ideas of same author about synthetic neighborhoods [21].

2.3.1 (DN) :

Let E be a metrisable locally convex space with an increasing fundamental system of seminorms $|| \ ||_1 \leq || \ ||_2 \leq \dots$. If there is a continuous norm $|| \ ||_0$ on E such that for all $k \in \mathbb{N}$ there exists $p \in \mathbb{N}$ and $C > 0$ such that

$$||x||_k^2 \leq C||x||_0||x||_p$$

holds for every $x \in E$, then we say that E has the **Dominating Norm Property**, or briefly (DN).

Theorem 2.3.1 *If E and F are two isomorphic locally convex spaces, then either $E$and F both have the property (DN) or neither E nor F has the property (DN). In other words, (DN) is a topological invariant.*

Theorem 2.3.2 *If E is a locally convex space with property (DN), then the property (DN) is inherited by all closed subspaces of E.*

Last two theorems related to (DN) are well-known in functional analysis. We refer to [8] for proofs.

2.3.2 (Ω) :

Let E be a metrisable locally convex space with a decreasing fundamental system of zero neighborhoods $U_1 \supset U_2 \supset \dots$. If for every $p \in \mathbb{N}$ there is a $q \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there is a j and a constant $C > 0$ so that the inclusion

$$U_q \subset Cr^j U_k + \frac{1}{r} U_p \text{ holds for every } r > 0,$$

then we say that E has property (Ω).

The following theorems are well-known;

Theorem 2.3.3 (Ω) *is a topological invariant, i.e. if E and F are isomorphic, then either both E and F have the property (Ω) or none of them has this property.*

Theorem 2.3.4 *If E is a locally convex space with property (Ω), then the property (Ω) is inherited by all quotient spaces of E.*

Theorem 2.3.5 *The Whitney space $\varepsilon(K)$ has property (Ω) for K any compact set on $\mathbb{R}$.*

The spaces s and $\Lambda_\infty(\alpha)$ have both of the properties (DN) and (Ω).

2.3.3 Diametral Dimension

Now let us define diametral dimension which is a topological invariant. We start with the following:
Let E be a locally convex space, U_p and U_q be absolutely convex subsets of E such that for some $\rho > 0$, $U_q \subset \rho U_p$. For L a subspace of E define

$$d(U_q, U_p; L) = \inf\{d \geq 0 : U_q \subset dU_p + L\}.$$

Then define

$$d_n(U_q, U_p) = \inf\{d(U_q, U_p; L) : L \subset E \text{ subspace such that } dim(L) \leq n\}$$

Then $d_n(U_q, U_p)$ is called the n^{th} **width** (or **Kolmogorov Diameter**) of U_q with respect to U_p. Let E be a Fréchet Space with a fundamental system of zero neighborhoods $(U_n)_{n\in\mathbb{N}}$. Then the set of all sequences $(\gamma_n)_{n=0}^\infty$ such that for all $p \in \mathbb{N}$ there exists a $q \in \mathbb{N}$ such that the product $\gamma_n \cdot d_n(U_q, U_p)$ converges to zero as n tends to infinity is called the **diametral dimension of E** and denoted by $\Gamma(E)$. This can be formulated as follows:

$$\Gamma(E) = \{(\gamma_n)_{n=0}^\infty : \forall p\ \exists q\ : \gamma_n \cdot d_n(U_q, U_p) \to 0 \text{ as } n \to \infty\}. \tag{2.6}$$

Note that, if E is a normed space, then $\Gamma(E) = c_0$. The following theorem says that diametral dimension is also a topological invariant:

Theorem 2.3.6 *If two locally convex spaces E and F are isomorphic to each other, then*

$$\Gamma(E) = \Gamma(F).$$

Converse is not always true, i.e. diametral dimension of E may be equal to diametral dimension of F while E and F are not isomorphic. Proof of (2.3.6) can be found in [11].

2.3.4 Topological Invariant β

Now let us define another topological invariant β and mention some of its basic properties about our subject:

Definition 2.3.7 *Let E be a locally convex space with a fundamental system of neighborhoods $(U_n)_{n\in\mathbb{N}}$. Then the counting function β is defined as follows:*

$$\beta(t) = \beta(U_p, U_q, t) = \min\{\dim L : t \cdot U_q \subset U_p + L\}, \;\; t > 0$$

where L is a linear subspace of E.

We have the following relation between diametral dimension and β which was given in [1] without any proof:

Theorem 2.3.8 *Let E be a Fréchet Space with a fundamental system of neighborhoods $(U_n)_{n\in\mathbb{N}}$. Then sequence $(\gamma_n)_{n\in\mathbb{N}}$ is in diametral dimension of E if and only if for all $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ and for all $C \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that $\beta(U_p, U_q, C\gamma_n) \leq n$ for $n \geq n_0$ or:*

$$(\gamma_n) \in \Gamma(E) \iff \forall p \; \exists q : \; \forall C \; \exists n_0 : \; \beta(U_p, U_q, C\gamma_n) \leq n \text{ for } n \geq n_0. \quad (2.7)$$

For proof of (2.3.8) we need the following lemma:

Lemma 2.3.9 *$\beta(t)$ is equal to the cardinality of the set $\{n : d_n(U_q, U_p) > \frac{1}{t}\}$, i.e.*

$$\beta(U_p, U_q, t) = |\{n : d_n(U_q, U_p) > \frac{1}{t}\}|$$

Proof: First observe that the n^{th} Kolmogorov diameter d_n has the following property (which follows from definition) $d_n(U_q, U_p) \leq d_{n-1}(U_q, U_p)$ for all $n \geq 1$. Now assume that the set $\{n : d_n(U_q, U_p) > \frac{1}{t}\}$ is not empty, then

$$|\{n : d_n(U_q, U_p) > \frac{1}{t}\}| = \max_{n\in\mathbb{N}}\{d_n(U_q, U_p) > \frac{1}{t}\} + 1 \tag{2.8}$$

because if for some $n \in \mathbb{N}$ $d_n(U_q, U_p) > \frac{1}{t}$, so is $d_0(U_q, U_p) > \frac{1}{t}$. Now let $N_1 \in \mathbb{N}$ be such that

$$d_{N_1}(U_q, U_p) \leq \frac{1}{t} \text{ and } d_{N_1-1}(U_q, U_p) > \frac{1}{t} \tag{2.9}$$

and let $N_2 \in \mathbb{N}$ be such that

$$N_2 = \beta(t) = \min\{\dim L : t \cdot U_q \subset U_p + L\}. \tag{2.10}$$

Then by (2.9), there exists a subspace L of E with $\dim L \leq N_1$ such that $d(U_q, U_p; L) \leq \frac{1}{t}$. Then;

$$\begin{aligned} d(U_q, U_p; L) &\leq \frac{1}{t} \\ \Rightarrow U_q &\subset \frac{1}{t}U_p + L \\ \Rightarrow tU_q &\subset U_p + L \end{aligned}$$

which gives that

$$\beta(U_p, U_q, t) \leq N_1$$

which implies that

$$N_1 \geq N_2. \tag{2.11}$$

Now (2.10) implies that there exists a subspace L of E with $\dim L = N_2$ and $t \cdot U_q \subset U_p + L$. Then;

$$\begin{aligned} t \cdot U_q &\subset U_p + L \\ \Rightarrow U_q &\subset \frac{1}{t} \cdot U_p + L \\ \Rightarrow \frac{1}{t} &\geq d(U_q, U_p; L) \end{aligned} \tag{2.12}$$

since $d(U_q, U_p; L) \geq d_{N_2}(U_q, U_p)$

$$\frac{1}{t} \geq d_{N_2}(U_q, U_p) \tag{2.13}$$

which implies that

$$N_2 \geq N_1. \tag{2.14}$$

Then (2.11) and (2.14) implies that

$$N_1 = N_2.$$

This completes the proof of lemma.

Proof of theorem: Now let $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma(E)$. Then for all p, there exists q such that $\gamma_n \cdot d_n(U_q, U_p) \to 0$ as $n \to \infty$, so for all $C \neq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\gamma_n \cdot d_n(U_q, U_p) < \frac{1}{|C|}$ which implies that $d_n(U_q, U_p) < \frac{1}{|C \cdot \gamma_n|}$ for all $n > n_0$ which is equivalent to saying that $\beta(|C \cdot \gamma_n|) \leq n$, i.e.

$$\beta(U_p, U_q, C\gamma_n) \leq n \text{ for } n \geq n_0.$$

Conversely, let for all p there exist q such that for all $C > 0$ there exists n_0 such that for all $n \geq n_0$, $\beta(U_q, U_p, C \cdot \gamma_n) \leq n$. Then, by lemma $d_n(U_q, U_p) \leq \frac{1}{|C \cdot \gamma_n|}$ which implies that $d_n(U_q, U_p) \cdot |\gamma_n| \leq \frac{1}{C}$. Since this is true for all C, we get that as $n \to \infty$, $d_n(U_q, U_p) \cdot \ |\gamma_n| \ \to 0$, so $(\gamma_n)_{n=0}^{\infty}$ is in diametral dimension of E. This completes the proof of the theorem.

We have the following theorem by Zahariuta [20]:

Theorem 2.3.10 *Let E and F be two Fréchet Spaces with the corresponding fundamental system of neighborhoods $(U_m)_{m\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ respectively. If E and F are isomorphic then for all p_1 there exists p, for all q there exists q_1 and for all $\epsilon > 0$ such that $\beta_F(V_{p_1}, V_{q_1}, \epsilon t) \leq \beta_E(U_p, U_q, t)$, $t > 0$. Or more formally :*

$$\forall p_1\ \exists p\ \forall q\ \exists q_1, \epsilon > 0 : \beta_F(V_{p_1}, V_{q_1}, \epsilon t) \leq \beta_E(U_p, U_q, t), \ t > 0. \tag{2.15}$$

For proof of (2.15) please see [20]. Just like the case about diametral dimension, converse is not always true. In other words for E and F Fréchet Spaces, (2.15) may hold while E and F are not isomorphic.

2.4 Cantor-Type Sets, Generalized Cantor Sets

We work on the following generalization of classical Cantor set. Consider the interval $I_0 = [0,1]$. Then let $(N_n)_{(n=1)}^{\infty}$ be a sequence of positive integers, and $(l_n)_{n=0}^{\infty}$ be a sequence of positive real numbers such that the following relation holds:

$$l_0 = 1,\ N_n \cdot l_n < l_{n-1} \text{ for all } n \geq 1.$$

Then at the n^{th} step we do the following:
Delete $N_1 \cdot N_2 \dots N_{n-1} \cdot (N_n - 1)$ open intervals of length $\frac{l_{n-1} - N_n \cdot l_n}{N_n - 1}$ from each interval such that all the remaining closed intervals have length l_n. Call the compact set derived at n^{th} step I_n and enumerate each closed intervals forming I_n as $I_{n,1}, I_{n,2}, \dots, I_{N_1 \cdot N_2 \dots N_n}$. Then let K be the infinite intersection of I_n's:

$$K = \bigcap_{n=1}^{\infty} I_n.$$

The resulting compact set is called a generalized Cantor set and we also denote K as $K(l_n, N_n)$. Also define the sequence $(\alpha_n)_{(n=1)}^{\infty}$ as $\alpha_n = \log_{l_n} l_{n+1}$, i.e. $l_{n+1} = l_n^{\alpha_n}$.

We will consider mainly the following special cases for K; $N_n = N$ for all n for some $N \in \mathbb{N}$ fixed or $N_n \nearrow \infty$ as $n \to \infty$. Then corresponding compact sets are denoted as K_N and K_∞ respectively. If in addition there is a constant α such that $\alpha_n = \alpha$ for all $n \in \mathbb{N}$, i.e. $l_n = l_{n-1}^{\alpha}$ for all $n \in \mathbb{N}$ then we denote these sets $K_N^{(\alpha)}$ and $K_\infty^{(\alpha)}$ respectively.
Let $\varepsilon(K)$ denote the space of Whitney functions on a Cantor-type set K. Each function $f \in \varepsilon(K)$ is extendable to a function in $C^\infty(\mathbb{R})$. If there exists an operator $T : \varepsilon(K) \to C^\infty(\mathbb{R})$ such that T is linear and continuous, then K is said to have **extension property**. Note that when considering Cantor-type sets, we will restrict ourselves to the case $l_n \leq \frac{l_{n-1} - N_n l_n}{N_n - 1}$ since otherwise K is uniformly perfect and it has the extension property. In [15] it was shown by Tidten that if $\varepsilon(K)$ has property (DN) then K has the extension property and vice versa. We have the following theorem due to Arslan, Goncharov and Kocatepe [1]:

Theorem 2.4.1 *If* $\liminf_{n\to\infty} \alpha_n > N$ *then the space* $\varepsilon(K_N)$ *does not have the property* (DN) *(or the set* K_N *does not have the extension property). If* $\limsup_{n\to\infty} \alpha_n < N$ *then the space* $\varepsilon(K)$ *has the property* (DN) *(or the set* K_N *has the extension property).*

For K_∞, we have the following theorem about extension property by same authors [1]:

Theorem 2.4.2 *The space* $\varepsilon(K)$ *has the property* (DN) *(or* K_∞ *has the extension property) if and only if there exists a constant* M *such that*

$$l_n \geq \left(\frac{l_{n-1} - N_n l_n}{N_n - 1}\right)^M \quad \textit{for all } n \in \mathbb{N}.$$

Proof of (2.4.1) and (2.4.2) can be found in [1].

We will use the notation "$\sim$" in the following sense:
For two functions F and G, $F \sim G$ means that for some $C,\ t_0$ we have

$$\frac{1}{C} F\left(\frac{t}{C}\right) \leq G(t) \leq C \cdot F(Ct), \quad \text{for all } t > t_0.$$

The following theorem by Arslan, Goncharov and Kocatepe [1] says that asymptotically if $t \sim l_n^{p-q}$, then $\beta(U_p, U_q, t) \sim N_1 \cdots N_n$ as $n \to \infty$. This theorem is of main importance in this thesis. So, for completeness, we will give its detailed proof.:

Theorem 2.4.3 *Let* $E = \varepsilon(K(l_n, N_n))$ *and* $p < q$ *be two fixed natural numbers. If* $t \leq \frac{1}{5} l_n^{p-q}$, *then* $\beta(U_p, U_q, t) \leq (q+1) N_1 \ldots N_n$. *If* $t \geq 5(q-p)! l_n^{p-q}$, *then* $\beta(U_p, U_q, t) \geq N_1 \ldots N_n$.

Proof: **Upper bound of** β

If for some subspace L we have $t \cdot U_q \subset U_p + L$, then $\beta(t) \leq \dim L$ (by definition of β). Let us fix n such that $5t \leq l_n^{p-q}$. Set $M = N_1 \ldots N_n$. Remember that while constructing Cantor-type sets, we denoted the set derived at the n^{th} step of construction with I_n and since $I_n = \bigcup_{1\le k\le M} I_{n,k}$ where $I_{n,k} = [a_k, b_k]$. For $k = 1, 2, \ldots, M$ and $j = 0, 1, \ldots, q$, let $e_{k,j}(x) = \frac{(x-a_k)^j}{j!}$ if $x \in K \cap I_{n,k}$ and $e_{k,j} = 0$ otherwise on K. We take $L = \text{Span}(e_{k,j})_{k=1,j=0}^{M,q}$. Then $\dim L = (q+1)M$ and it is enough to show that for any function f with $||f||_q \le t$ there exists a function $g \in L$ such that

$$||f-g||_p \le 1. \tag{2.16}$$

Given $f \in tU_q$ let

$$g = \sum_{k=1}^{M}\sum_{j=0}^{q} f^{(j)}(a_k)\cdot e_{k,j}.$$

Clearly, if $x \in I_{n,k}$, then $(f-g)(x) = R^q_{a_k} f(x)$. Since $|x-a_k| \le L_n$ we get

$$|(f-g)^{(i)}(x)| \le ||f||_q \cdot |x-a_k|^{q-i} \le t\cdot l_n^{q-i},\ i\le p. \tag{2.17}$$

Let now

$$A_p = \frac{|\left(R^p_y(f-g)\right)^{(i)}(x)|}{|x-y|^{p-i}},\ x,y\in\ K,\ x\ne y,\ i\le p.$$

If $x, y \in I_{n,k}$ for some k, then

$$\begin{aligned} R^p_y(f-g)(x) &= f(x) - g(x) - T^p_y(R^q_0 f)(x) \\ &= T^q_y f(x) + R^q_y f(x) - T^q_0 f(x) - T^p_y(R^q_0 f)(x) \end{aligned}$$

Now the following identity which is proved in [7]:

$$T^q_y f(x) - T^q_a f(x) = T^q_y(R^q_a f)(x)$$

implies that ;

$$T^q_y f(x) - T^q_0 f(x) = T^q_y(R^q_0 f)(x)$$

and so we can use the following representation:

$$R^p_y(f-g)(x) = R^q_y f(x) + \sum_{m=p+1}^{q}\left(R^q_{a_k} f\right)^{(m)}\cdot\frac{(x-y)^m}{m!}.$$

Therefore

$$A_p \le ||f||_q \cdot |x-y|^{i-p} \cdot |x-y|^{i-p} + \sum_{p+1}^{q} ||f||_q \cdot |y-a_k|^{q-m} \cdot \frac{|x-y|^{m-i}}{(m-i)!} \cdot |x-y|^{i-p}.$$

As $|x-y| \le l_n$,

$$A_p \le t \cdot l_n^{q-p} \left(1 + \sum \frac{1}{(m-i)!}\right) < (e+1)\, t \cdot l_n^{q-p}.$$

On the other hand, if the points $x,\ y$ are located on different subintervals of I_n, (i.e. $x \in I_{n,k_1}$ and $y \in I_{n,k_2}$ but $k_1 \neq k_2$) then

$$|x-y| > \frac{l_{n-1} - N_n l_n}{N_n - 1} > l_n$$

by assumption. For this case

$$A_p \le |\left(f-g\right)^{(i)}(x)| \cdot |x-y|^{i-p} + \sum_{m=i}^{p} \frac{|\left(f-g\right)^{(m)}(y)|}{(m-i)!} |x-y|^{m-p}.$$

From (2.17) it follows that

$$A_p \le t \cdot l_n^{q-p} + t \cdot l_n^{q-p} \sum_{m=i}^{p} \frac{1}{(m-i)!}.$$

So,

$$||f-g||_p \le t \cdot l_n^{q-p} (2+e) < 1,$$

i.e.

$$||f-g||_p < 1$$

as desired, so

$$\beta(t) \le \dim L = (q + 1 N_1 \cdots N_n).$$

Lower bound of β

We use Tikhomirov theorem ([16] or [9]) to get a lower estimate for Kolmogorov

diameters:
If $d \cdot U_p \cap L \subset U_q$ with $\dim L = n+1$, then $d_n\left(U_q, U_p\right) \geq d$.

Therefore,

$$\beta\left(t\right) \geq \dim L \text{ if } U_p \cap L \subset \left(1-\epsilon_0\right) t \cdot U_q \text{ withsome } \epsilon_0 > 0.$$

Let us take

$$L = \operatorname{Span}\left(e_{k,q}\right)_{k=1} M$$

and fix

$$f = \sum_{k=1}^{M} C_k \cdot e_{k,q} \in L \cap U_p.$$

Since

$$1 \geq ||f||_p \geq |f^{(p)}\left(b_k\right)| \geq |C_K| \frac{l_n^{q-p}}{(q-p)!},$$

we have

$$|C_k| \leq (q-p)! l_n^{p-q} \text{ forall } k.$$

Let $x \in I_{n,k}$. Then

$$|f^{(i)}\left(x\right)| \leq |C_k| \frac{l_n^{q-i}}{(q-i)!}, \; i \leq q,$$

hence $|f|_q \leq (q-p)! l_n^{p-q}$.

if $x, y \in I_{n,k}$ then $R_y^q f\left(x\right) = 0$. Otherwise, $|x-y| \geq \frac{l_{n-1} - N_n l_n}{N_n - 1} \geq l_n$ and arguing as before, we obtain

$$\begin{aligned}
|\left(R_y^q f\right)^{(i)}\left(x\right)| \cdot |x-y|^{i-q} &\leq |f^{(i)}\left(x\right)| \cdot h_n^{i-q} + \sum_{m=i}^{q} \frac{f^{(m)(y)}}{(m-i)!} h_n^{m-q} \\
&\leq \frac{|C_k|}{(q-p)!} + \sum_{m=i}^{q} \frac{|C_{k_1}|}{(m-i)!} \\
&\leq (q-p)! l_n^{p-q}\left(1+e\right).
\end{aligned}$$

If we take ϵ_0 with $2+e<5\,(1-\epsilon_0)$, then

$$||f||_q \leq (q-p)!l_n^{p-q}\,(2+e) < t\cdot(1-\epsilon_0)$$

and

$$\beta\,(t) \geq \dim L = M = N_1 \ldots N_n.$$

$\square$

Now when does the space $\varepsilon\,(K\,(l_n, N_n))$ has the property (Ω)? It is a well known fact that if $K \subset \mathbb{R}$ is a perfect compact set then

$$\varepsilon\,(K) \simeq C^{\infty}\,(L)\,/Z$$

where L is a compact interval in $\mathbb{R}$ which contains K and $Z = \{f \in C^{\infty}\,(L)\ : f\mid_K \equiv 0\}$. It was shown in [9] by Mityagin that $C^{\infty}\,(L) \simeq s$. Hence $\varepsilon\,(K)$ is isomorphic to a quotient space of s, the space of rapidly decreasing sequences. Since s has property (Ω), $\varepsilon\,(K)$ has the property (Ω). It also follows that $\varepsilon\,(K\,(l_n, N_n))$ is nuclear, since s is nuclear and nuclearity is inherited by quotient spaces.

Chapter 3

Existence of Basis in Some Whitney Spaces

Existence of basis in nuclear Fréchet spaces has aroused many researchers interest up to now. It has been shown by Mityagin and Zobin [10] that there are nuclear Fréchet spaces without basis. However it is still not known whether there is a nuclear function space without a basis. It is believed that if there is such a space, it will be found among Whitney spaces. In this chapter, we show that under some restrictions, Whitney spaces on Cantor-type compact sets have basis. For both of the following theorems, first we will give a simpler proof demonstrating the basic idea and later a detailed proof will be given. We start with the following theorem due to A. Aytuna, J. Krone and Terzioğlu [2] which plays an important role in this thesis :

Theorem 3.0.4 *Let E be a nuclear Frèchet space with property (DN) and (Ω). Let $\Gamma(E) = \Gamma(\Lambda_\infty(\beta))$ for some $\beta = (\beta_n)_{n\in\mathbb{N}}$. If*

$$\sup_n \frac{\beta_{2n}}{\beta_n} < \infty \tag{3.1}$$

holds, then E has a basis.

Remark 3.0.5 *Note that it was shown by Terzioğlu in [13] that if E is a Fréchet space with property (DN) and (Ω), then there is $\beta = (\beta_n)_{n\in\mathbb{N}}$ such that $\Gamma(E) =$*

$\Gamma(\Lambda_\infty(\beta))$, and β is unique up to equivalence, i.e. if $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}}$ are two such sequences, then there is a constant $C > 0$ such that $\frac{1}{C} \leq \frac{\alpha_n}{\beta_n} \leq C$ for all n.

Proof: By Lemma 2.3 in [2], if E is a nuclear space with properties (DN) and (Ω) and if $\Gamma(E) \subset \Gamma(\Lambda_\infty(\beta))$, then there exists a local imbedding of $\Lambda_\infty(\beta)$ into E.

By Corollary 1.5 in [2], if $\Lambda_\infty(\beta)$ is nuclear and stable (stability of $\Lambda_\infty(\beta)$ is equivalent to (3.1)) and if E is a Fréchet space with (DN) and (Ω), and $\Gamma(\Lambda_\infty(\beta)) \subset \Gamma(E)$ holds, and there exists a local imbedding of $\Lambda_\infty(\beta)$ into E then E is isomorphic to $\Lambda_\infty(\beta)$, i.e.;

$$E \simeq \Lambda_\infty(\beta).$$

So existence of basis in E depends on existence of basis in $\Lambda_\infty(\beta)$. We know that $\Lambda_\infty(\beta)$ always has a basis since it's a Köthe space. This completes the proof of theorem.

$\square$

Theorem 3.0.6 *Let $K_N = K(l_n, N_n)$ be a generalized Cantor-type set where $N_n = N$ fixed for all $n \in \mathbb{N}$, $E = \varepsilon(K)$ be the space of infinitely differentiable Whitney functions defined on K. If*

$$\limsup_{n\to\infty} \alpha_n < N$$

then the space E has a basis.

Proof:

Remark 3.0.7 *By 2.4.1 if $\limsup_n \alpha_n < N$ then E has (DN).*

Since $l_n = l_{n-1}^{\alpha_{n-1}}$, define $A(n) = \alpha_1 \ldots \alpha_n,\ n \in \mathbb{N}$ and extend it to interval $(n, n+1)$ linearly (or any other way so that $A(x)$ is increasing). Without loss of generality, we may assume that $l_1 = \frac{1}{e}$. Then;
$t \sim l_n^{p-q} = e^{\alpha_1 \ldots \alpha_n (q-p)} = e^{A(n)(q-p)} \Rightarrow \beta(t) \sim \underbrace{N \ldots N}_{n\ times\ N} = N^n$.
Then $\ln t \sim A(n) . (q-p) \Rightarrow A(n) \sim \frac{\ln t}{q-p} \Rightarrow n \sim A^{-1}\left(\frac{\ln t}{q-p}\right)$ so

$$\beta(U_q, U_p, t) \sim N^n = N^{A^{-1}\left(\frac{\ln t}{q-p}\right)}.$$

Remember that, (2.3.8) says : $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma(\varepsilon(K)) \Leftrightarrow \forall\ p\ \exists\ q\ : \forall\ C \ \exists\ n_0 :$ $\beta(U_p, U_q, C \cdot \gamma_n) \leq\ n\ \forall\ n\ \geq n_0$.

Now let $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma(\varepsilon(K))$, then $\forall\ p\ \exists\ q\ \forall\ C\ \exists\ n_0$:

$$\begin{aligned} \beta(C \cdot \gamma_n) &\leq n \Rightarrow \\ N^{A^{-1}\left(\frac{\ln(C\cdot\gamma_n)}{q-p}\right)} &\leq n \end{aligned} \tag{3.2}$$

for all $n \geq n_0$ which implies that

$$\begin{aligned} A^{-1}\left(\frac{\ln(C \cdot \gamma_n)}{q-p}\right) &\leq \frac{\ln n}{\ln N} \Rightarrow \\ \ln(C \cdot \gamma_n) &\leq (q-p) \cdot A\left(\frac{\ln n}{\ln N}\right) \Rightarrow \\ C \cdot \gamma_n &\leq e^{(q-p).A\left(\frac{\ln n}{\ln N}\right)} \end{aligned}$$

for all $n \geq n_0$. Then letting $q - p = M$ and $\beta_n = A\left(\frac{\ln n}{\ln N}\right)$, we have

$$\gamma_n . e^{-M.\beta_n} \leq \frac{1}{C} \tag{3.3}$$

for all $n \geq n_0$. So

$$(\gamma_n)_{n\in\mathbb{N}} \in \Gamma(\varepsilon(K)) \Leftrightarrow \exists M > 0 \text{ such that } \gamma_n . e^{-M.\beta_n} \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Thus

$$\Gamma\left(\varepsilon\left(K\right)\right)=\Lambda_{\infty}{}'\left(\beta\right)=\Gamma\left(\Lambda_{\infty}\left(\beta\right)\right) \tag{3.5}$$

where $\beta=\left(\beta_n\right)_{n\in\mathbb{N}}$.

Now let us show stability of β:

Assume

$$k\leq\left(\frac{\ln n}{\ln N}\right)<k+1 \text{ for } k\in\mathbb{N} \tag{3.6}$$

Then

$$\frac{\ln\left(2n\right)}{lnN}=\underbrace{\frac{\ln 2}{\ln N}}_{between\ 0\ and\ 1(since\ 2\leq N)}+\underbrace{\frac{\ln n}{\ln N}}_{<1}<1+\left(k+1\right)=k+2 \tag{3.7}$$

So;

$$k\leq\frac{\ln\left(2n\right)}{\ln N}<k+2 \tag{3.8}$$

Then

$$\begin{aligned}(3.6)\Rightarrow&\quad \alpha_1\ldots\alpha_k\leq\beta_n<\alpha_1\ldots\alpha_{k+1}\\ (3.8)\Rightarrow&\quad \alpha_1\ldots\alpha_k\leq\beta_{2n}<\alpha_1\ldots\alpha_{k+2}.\end{aligned}$$

So;

$$1\leq\frac{\beta_{2n}}{\beta_n}<\frac{\alpha_1\ldots\alpha_{k+2}}{\alpha_1\ldots\alpha_k}=\alpha_{k+1}\cdot\alpha_{k+2}. \tag{3.9}$$

Since $\left(\alpha_n\right)_{n\in\mathbb{N}}$ is bounded, then $\frac{\beta_{2n}}{\beta_n}$ is also bounded, so β is stable, then by (3.0.4), $E\simeq\Lambda_{\infty}(\beta)$. Since $\Lambda_{\infty}(\beta)$ has a basis, E has also a basis.

Remark 3.0.8 *For simplicity we are using $\sim$ notation in the proof. This may make careful reader a bit suspicious about reliability of proof. For detailed proof one can see later pages.*

Theorem 3.0.9 *Let $K_\infty = K(l_n, N_n)$ be a generalized Cantor-type set where $\lim_{n\to\infty} N_n = \infty$, $E = \varepsilon(K_\infty)$ be the space of infinitely differentiable Whitney functions defined on K_∞ and assume that E has property (DN). If*

$$\limsup_{n\to\infty} \alpha_n < \infty$$

then the space E has a basis.

Proof:

Let $K = K_\infty$, $N_n = N^{T_n}$, where

$$N = \min_i \left\{ N_i : N_i > \limsup_{n\to\infty} \alpha_n \right\},$$

$T_n \nearrow \infty$, so $N_1 \cdot N_2 \dots N_n = N^{T_1+T_2+\dots+T_n}$.
Now define $S(n) = T_1 + T_2 + \dots + T_n$, $n \in \mathbb{N}$. $S(n) \nearrow \infty$ and extend $S(n)$ to the interval $(n, n+1)$ linearly (or any other way so that $S(x)$ is increasing).
Define $A(n)$ and extend it to $A(x)$ as before.
$t \sim l_n^{p-q} = e^{\alpha_1 \dots \alpha_n (q-p)} = e^{A(n)(q-p)} \Rightarrow \beta(t) \sim N_1 \dots N_n$.
Then $\ln t \sim A(n) \cdot (q-p) \Rightarrow A(n) \sim \frac{\ln t}{q-p} \Rightarrow n \sim A^{-1}\left(\frac{\ln t}{q-p}\right)$ so

$$\beta(U_q, U_p, t) \sim N_1 \dots N_n = N_1 \dots N_{A^{-1}\left(\frac{\ln t}{q-p}\right)} = N^{S\left(A^{-1}\left(\frac{\ln t}{q-p}\right)\right)}.$$

Remember that (2.3.8) says that : $(\gamma_n) \in \Gamma(\varepsilon(K)) \Leftrightarrow \forall\ p\ \exists\ q\ : \forall\ C\ \exists\ n_0 :$ $\beta(U_p, U_q, C\cdot\gamma_n) \leq\ n\ \forall\ n\ \geq n_0$.
Now let $(\gamma_n) \in \Gamma(\varepsilon(K))$, then $\forall\ p\ \exists\ q\ \forall\ C\ \exists\ n_0 :$

$$\begin{array}{ccc} \beta(C\cdot\gamma_n) & \leq & n \Rightarrow \\ N^{S\left(A^{-1}\left(\frac{\ln(C\cdot\gamma_n)}{q-p}\right)\right)} & \leq & n \end{array}$$

for all $n \geq n_0$ which implies that

$$\begin{aligned}
S\left(A^{-1}\left(\frac{\ln\left(C\cdot\gamma_n\right)}{q-p}\right)\right) &\leq \frac{\ln n}{\ln N} \Rightarrow \\
A^{-1}\left(\frac{\ln\left(C\cdot\gamma_n\right)}{q-p}\right) &\leq S^{-1}\left(\frac{\ln n}{\ln N}\right) \Rightarrow \\
\ln\left(C\cdot\gamma_n\right) &\leq \left(q-p\right).A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right) \Rightarrow \\
C\cdot\gamma_n &\leq e^{(q-p).A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right)}
\end{aligned}$$

for all $n \geq n_0$. Then letting $q-p = M$ and $\beta_n = A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right)$, we have $\gamma_n.e^{-M.\beta_n} \leq \frac{1}{C}$ for all $n \geq n_0$. So

$$\left(\gamma_n\right)_{n\in\mathbb{N}} \in \Gamma\left(\varepsilon\left(K\right)\right) \Leftrightarrow \exists M > 0 \text{ such that } \gamma_n.e^{-M.\beta_n} \to 0 \text{ as } n \to \infty.$$

Thus $\Gamma\left(\varepsilon\left(K\right)\right) = {\Lambda_\infty}'\left(\beta\right) = \Gamma\left(\Lambda_\infty\left(\beta\right)\right)$ where $\beta = \left(\beta_n\right)_{n\in\mathbb{N}}$.

Now let us show stability of β:

Assume

$$k \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) < k+1 fork \in \mathbb{N} \tag{3.10}$$

Then $S^{-1}\left(\frac{\ln 2n}{lnN}\right) = S^{-1}\left(\underbrace{\frac{\ln 2}{\ln N}}_{<1} + \frac{\ln n}{\ln N}\right) < 1 + (k+1)$ for sufficiently large n (assuming linearity of S^{-1} gives same result), i.e.;

$$k \leq S^{-1}\left(\frac{\ln 2n}{\ln N}\right) < k+2 \tag{3.11}$$

Then;

$$\begin{aligned}
(3.10) \;\Rightarrow\quad \alpha_1 \ldots \alpha_k \leq\; & \beta_n < \alpha_1 \ldots \alpha_k \cdot \alpha_{k+1} \\
(3.11) \;\Rightarrow\quad \alpha_1 \ldots \alpha_k \leq\; & \beta_{2n} < \alpha_1 \ldots \alpha_{k+1} \cdot \alpha_{k+2}
\end{aligned}$$

So;

$$1 \leq \frac{\beta_{2n}}{\beta_n} < \alpha_{k+1} \cdot \alpha_{k+2}$$

If $(\alpha_n)_{n\in\mathbb{N}}$ is bounded, say $\alpha_n \leq B$ then $1 \leq \frac{\beta_{2n}}{\beta_n} \leq B^2$, so $(\beta_n)_{n\in\mathbb{N}}$ is stable. Then by (3.0.4), $E \simeq \Lambda_\infty(\beta)$. Since $\Lambda_\infty(\beta)$ has a basis, E has also a basis.

Remark 3.0.10 *For simplicity we are using $\sim$ notation in the proof. This may make careful reader a bit suspicious about reliability of proof. For detailed proof one can see later pages.*

Detailed proof of (3.0.6):

Let $p \leq q$ be fixed. Then

$$0 \leq \frac{1}{l_1^{q-p}} \leq \frac{1}{l_2^{q-p}} \leq \ldots \leq \frac{1}{l_{n-1}^{q-p}} \leq \frac{1}{l_n^{q-p}} \leq \ldots \text{ and } \frac{1}{l_n^{q-p}} \rightarrow \infty \text{ as } n \rightarrow \infty \tag{3.12}$$

Find n_0 such that for all $n \geq n_0$

$$5\,(q-p)!\frac{1}{l_{n-2}^{q-p}} \leq \frac{1}{l_{n-1}^{q-p}} \text{ and } \frac{1}{l_n^{p-q}} \leq \frac{1}{5}\frac{1}{l_{n+1}^{q-p}}. \tag{3.13}$$

Let $t > 0$ be large enough (say $t \geq \frac{1}{l_{n_0-1}^{q-p}}$). Find a unique $n \geq n_0$ s.t.

$$\frac{1}{l_{n-1}^{q-p}} \leq t \leq \frac{1}{l_n^{p-q}} \tag{3.14}$$

Then by (3.13)

$$5\,(q-p)!\frac{1}{l_{n-2}^{q-p}} \leq t \leq \frac{1}{5}\frac{1}{l_{n+1}^{q-p}} \tag{3.15}$$

On the other hand, (3.14) holds if and only if

$$\begin{array}{rcl} e^{(q-p)\alpha_1\ldots\alpha_{n-1}} \overset{(1)}{\leq} & t & \overset{(2)}{\leq} e^{(q-p)\alpha_1\ldots\alpha_n} \\ \Leftrightarrow (q-p)\,A\,(n-1) \overset{(1)}{\leq} & \ln t & \overset{(2)}{\leq} (q-p)\,A\,(n) \\ \Leftrightarrow A\,(n-1) \overset{(1)}{\leq} & \frac{\ln t}{q-p} & \overset{(2)}{\leq} A\,(n) \\ \Leftrightarrow n-1 \overset{(1)}{\leq} & A^{-1}\left(\frac{\ln t}{q-p}\right) & \overset{(2)}{\leq} n \\ \Leftrightarrow A^{-1}\left(\dfrac{\ln t}{q-p}\right) \overset{(1)}{\leq} & n & \overset{(2)}{\leq} A^{-1}\left(\dfrac{\ln t}{q-p}\right)+1 \end{array}$$

(which is equivalent to 3.14.)

Now (3.15) $\Rightarrow$

$$N^{n-2} \overset{(a)}{\leq} \beta\,(U_p, U_q, t) \overset{(b)}{\leq} (q+1)\,N^{n+1}. \tag{3.16}$$

By (3.14) ;

$$N^{A^{-1}\left(\frac{\ln t}{q-p}\right)-2} \overset{(a,2)}{\leq} \beta\left(U_p, U_q, t\right) \overset{(b,1)}{\leq} (q+1)\cdot N^{A^{-1}\left(\frac{\ln t}{q-p}\right)+2} \text{ for } t \geq t_0. \tag{3.17}$$

Let now $(\gamma_n) \in \Gamma\left(\varepsilon\left(K\right)\right)$. Then $\forall p\ \exists q\ \forall C\ \exists n_1$ s.t. $\beta\left(U_p, U_q, C\cdot\gamma_n\right) \leq n$ for $n \geq n_1$. Assume $n \geq n_1$ and $C\cdot\gamma_n \geq t_0$. Then (a,2) $\Rightarrow$

$$\begin{aligned}
N^{A^{-1}\left(\frac{\ln(C\cdot\gamma_n)}{q-p}\right)-2} &\leq n \Rightarrow \\
A^{-1}\left(\frac{\ln\left(C\cdot\gamma_n\right)}{q-p}\right)-2 &\leq \frac{\ln n}{\ln N} \Rightarrow \\
A^{-1}\left(\frac{\ln\left(C\cdot\gamma_n\right)}{q-p}\right) &\leq \frac{\ln n}{\ln N}+2 \Rightarrow \\
\ln\left(C\cdot\gamma_n\right) &\leq (q-p)\cdot A\left(\frac{\ln n}{\ln N}+2\right) \Rightarrow \\
C\cdot\gamma_n &\leq e^{(q-p)\cdot A\left(\frac{\ln n}{\ln N}\right)+2}
\end{aligned}$$

Let $\beta_n = A\left(\frac{\ln n}{\ln N}\right)$. Given n, choose an integer $k = k(n)$ s.t.

$$\begin{aligned}
k \leq &\quad \frac{\ln n}{\ln N} &< k+1 \Rightarrow \\
k+2 \leq &\quad \frac{\ln n}{\ln N}+2 &< k+3 \Rightarrow \\
A\left(k+2\right) \leq &\quad A\left(\frac{\ln n}{\ln N}+2\right) &< A\left(k+3\right)
\end{aligned}$$

which is equivalent to the following:

$$\alpha_1\ldots\alpha_k\cdot\alpha_{k+1}\cdot\alpha_{k+2} \leq \quad A\left(\frac{\ln n}{\ln N}+2\right) \quad < \alpha_1\ldots\alpha_{k+2}\cdot\alpha_{k+3}$$

and

$$A\left(k\right) \leq \quad A\left(\frac{\ln n}{\ln N}\right) \quad < A\left(k+1\right)$$

i.e.

$$\alpha_1\ldots\alpha_k \leq \qquad \beta_n \qquad < \alpha_1\ldots\alpha_k\cdot\alpha_{k+1}.$$

Thus;

$$1 < \frac{A\left(\frac{\ln n}{\ln N}+2\right)}{\beta_n} < \frac{\alpha_1 \dots \alpha_{k+3}}{\alpha_1 \dots \alpha_k}$$

i.e.

$$1 < \frac{A\left(\frac{\ln n}{\ln N}+2\right)}{\beta_n} < \alpha_{k+1} \cdot \alpha_{k+2} \cdot \alpha_{k+3}.$$

Since (α_n) is bounded, the sequences $\left(A\left(\frac{\ln n}{\ln N}+2\right)\right)$ and (β_n) are equivalent. Thus $\exists\ M$ such that $\gamma_n.e^{-M\cdot\beta_n} \to 0$ as $n \to \infty$ so;

$$(\gamma_n) \in \Gamma\left(\Lambda_\infty(\beta)\right) = \Lambda_\infty(\beta)'$$

Conversely assume $(\gamma_n)_n \in \Gamma\left(\Lambda_\infty(\beta)\right) = \Lambda_\infty(\beta)'$ where $\beta_n = A\left(\frac{\ln n}{\ln N}\right)$. Obviously, for all $C > 0$, the sequence $(C \cdot \gamma_n)_n$ also is in $\Gamma\left(\Lambda_\infty(\beta)\right)$. Then $\exists$ $M > 0$ s.t. $\lim_{n\to\infty} |C \cdot \gamma_n|.e^{-M.\beta_n} = 0$ or $|C \cdot \gamma_n| \leq e^{M.\beta_n}$ for large n. Show that $(C \cdot \gamma_n) \in \Gamma\left(\varepsilon(K)\right)$, i.e. $\forall p\ \exists q : \forall C > 0, \exists n_0\ \beta\left(U_p, U_q, C \cdot \gamma_n\right) \leq n$, for all $n > n_0$.

We find α such that $\limsup_n \alpha_n < \alpha < N$. Then $\alpha_n \leq \alpha$ for large n. Given n, let $k = k(n)$ be such that $k \leq \frac{\ln n}{\ln N} < k+1$. Since $\lim_{q\to\infty} \frac{q}{(q+1)^{\frac{\ln\alpha}{\ln N}}} = +\infty$, we have an integer q_0 s.t. $\forall q \geq q_0$

$$2M\alpha^4 \leq \frac{q}{(q+1)^{\frac{\ln\alpha}{\ln N}}}. \tag{3.18}$$

Given p, choose q s.t. $q \geq 2p$ and $q \geq q_0$. Then $q/2 \leq q - p$ and (3.18) holds. Since $(q+1)^{\frac{\ln\alpha}{\ln N}} = \alpha^{\frac{\ln(q+1)}{\ln N}}$, (3.18) $\Leftrightarrow$

$$M.\alpha^{4+\frac{\ln(q+1)}{\ln N}} < q/2$$

Choose an integer μ s.t. $N^\mu \leq q+1 < N^{\mu+1}$. Then $\mu \leq \frac{\ln(q+1)}{\ln N}$. So (3.18) $\Rightarrow$

$$M \cdot \alpha^{4+\mu} \leq M \cdot \alpha^{4+\frac{\ln(q+1)}{\ln N}} \leq q/2 \leq q - p \Rightarrow$$

$$\begin{aligned} M \cdot \alpha_{k-\mu-2} \dots \alpha_{k+1} &\leq q-p \\ \Rightarrow M \cdot \alpha_1 \dots \alpha_{k+1} &\leq (q-p) \cdot \alpha_1 \dots \alpha_{k-\mu-3} \\ &\text{i.e.} \\ M \cdot A(k+1) &\leq (q-p) \cdot A(k-\mu-3). \end{aligned}$$

Since $\frac{\ln(q+1)}{\ln N} < \mu+1$, we have $k-\mu-3 = k-(\mu+1)-2 \leq \frac{\ln n - \ln(q+1)}{\ln N} - 2$. It follows that

$$\begin{aligned} M \cdot \underbrace{A\left(\frac{\ln n}{\ln N}\right)}_{\beta_n} &\leq (q-p) \cdot A\left(\frac{\ln n - \ln(q+1)}{\ln N} - 2\right) \\ \Rightarrow |C \cdot \gamma_n| &\leq e^{(q-p)\cdot A\left(\frac{\ln n - \ln(q+1)}{\ln N} - 2\right)} \Rightarrow \\ \frac{\ln |C \cdot \gamma_n|}{q-p} &\leq A\left(\frac{\ln n - \ln(q+1)}{\ln N} - 2\right) \Rightarrow \\ A^{-1}\left(\frac{\ln |C \cdot \gamma_n|}{q-p}\right) &\leq \frac{\ln n - \ln(q+1)}{\ln N} - 2 \Rightarrow \\ A^{-1}\left(\frac{\ln |C \cdot \gamma_n|}{q-p}\right) + 2 &\leq \frac{\ln n - \ln(q+1)}{\ln N} \Rightarrow \\ N^{A^{-1}\left(\frac{\ln |C\cdot\gamma_n|}{q-p}\right)+2} &= e^{\left(A^{-1}\left(\frac{\ln |C\cdot\gamma_n|}{q-p}\right)+2\right)\cdot \ln N} \leq \\ e^{\ln n - \ln(q+1)} &= \frac{n}{q+1} \Rightarrow \\ (q+1) \cdot N^{A^{-1}\left(\frac{\ln |C\cdot\gamma_n|}{q-p}\right)+2} &\leq n. \end{aligned}$$

So by (3.17) we get $\beta(U_p, U_q; |C \cdot \gamma_n|) \leq n$ for n large enough.

Detailed proof of (3.0.9):

Let $p \leq q$ be fixed. Then

$$0 \leq \frac{1}{l_1^{q-p}} \leq \frac{1}{l_2^{q-p}} \leq \ldots \leq \frac{1}{l_{n-1}^{q-p}} \leq \frac{1}{l_n^{q-p}} \leq \ldots$$

$$\text{and } \frac{1}{l_n^{q-p}} \to \infty \text{ as } n \to \infty$$

Find n_0 such that for all $n \geq n_0$

$$5\,(q-p)!\frac{1}{l_{n-2}^{q-p}} \leq \frac{1}{l_{n-1}^{q-p}} \text{ and } \frac{1}{l_n^{p-q}} \leq \frac{1}{5}\frac{1}{l_{n+1}^{q-p}} \tag{3.19}$$

Let $t > 0$ be large enough (say $t \geq \frac{1}{l_{n_0-1}^{q-p}}$). Find a unique $n \geq n_0$ s.t.

$$\frac{1}{l_{n-1}^{q-p}} \overset{(1)}{\leq} t \overset{(2)}{\leq} \frac{1}{l_n^{p-q}} \tag{3.20}$$

Then by (3.19)

$$5\,(q-p)!\frac{1}{l_{n-2}^{q-p}} \leq t \leq \frac{1}{5}\frac{1}{l_{n+1}^{q-p}} \tag{3.21}$$

On the other hand, (3.20) holds if and only if

$$\begin{array}{rcl} e^{(q-p)\alpha_1\ldots\alpha_{n-1}} \leq & t & \leq e^{(q-p)\alpha_1\ldots\alpha_n} \\ \Leftrightarrow (q-p)\,A\,(n-1) \leq & \ln t & \leq (q-p)\,A\,(n) \\ \Leftrightarrow A\,(n-1) \leq & \frac{\ln t}{q-p} & \leq A\,(n) \\ \Leftrightarrow n-1 \leq & A^{-1}\left(\frac{\ln t}{q-p}\right) & \leq n \\ \Leftrightarrow A^{-1}\left(\dfrac{\ln t}{q-p}\right) \overset{(2)}{\leq} & n & \overset{(1)}{\leq} A^{-1}\left(\dfrac{\ln t}{q-p}\right) + 1 \end{array}$$

which is equivalent to (3.20).

First define

$$N = \min_i \left\{ N_i : N_i > \limsup_{n\to\infty} \alpha_n \right\},$$

and let α be any real number s.t. $N > \alpha > \limsup_{n\to\infty} \alpha_n$. Now (3.21) $\Rightarrow$

$$\begin{array}{rcl} N_1 \dots N_{n-2} & \overset{(a)}{\leq} \beta(U_p, U_q, t) \overset{(b)}{\leq} & (q+1) N_1 \dots N_n \cdot N_{n+1} \\ \text{i.e. } N^{S(n-2)} & \overset{(a)}{\leq} \beta(U_p, U_q, t) \overset{(b)}{\leq} & (q-1) . N^{S(n+1)} \end{array}$$

By (3.20) ;

$$N^{S\left(A^{-1}\left(\frac{\ln t}{q-p}\right)-2\right)} \overset{(a,2)}{\leq} \beta(t) \overset{(b,1)}{\leq} (q+1) \cdot N^{S\left(A^{-1}\left(\frac{\ln t}{q-p}\right)+2\right)} t \geq t_0 \tag{3.22}$$

Let now $(\gamma_n) \in \Gamma(\varepsilon(K))$. Then $\forall p \exists q \forall C \exists n_1$ such that $\beta(U_p, U_q, C \cdot \gamma_n) \leq n$ for $n \geq n_1$. Assume $n \geq n_1$ and $C \cdot \gamma_n \geq t_0$. Then (a,2) $\Rightarrow$

$$\begin{array}{rcl} N^{S\left(A^{-1}\left(\frac{\ln(C\cdot\gamma_n)}{q-p}\right)-2\right)} & \leq & n \Rightarrow \\ S\left(A^{-1}\left(\dfrac{\ln(C\cdot\gamma_n)}{q-p}\right)-2\right) & \leq & \dfrac{\ln n}{\ln N} \Rightarrow \\ A^{-1}\left(\dfrac{\ln(C\cdot\gamma_n)}{q-p}\right) & \leq & S^{-1}\left(\dfrac{\ln n}{\ln N}\right)+2 \Rightarrow \\ \ln(C\cdot\gamma_n) & \leq & (q-p)\cdot A\left(S^{-1}\left(\dfrac{\ln n}{\ln N}\right)+2\right) \Rightarrow \\ C\cdot\gamma_n & \leq & e^{(q-p)\cdot A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+2\right)} \end{array}$$

Let $\beta_n = A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right)$. Given n, choose an integer k such that;

$$\begin{array}{rcl} k \leq & S^{-1}\left(\frac{\ln n}{\ln N}\right) & < k+1 \Rightarrow \\ k+2 \leq & S^{-1}\left(\frac{\ln n}{\ln N}\right)+2 & < k+3 \Rightarrow \\ A(k+2) \leq & A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+2\right) & < A(k+3) \\ & \text{,i.e.} & \\ \alpha_1 \dots \alpha_{k+1} \cdot \alpha_{k+2} \leq & A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+2\right) & < \alpha_1 \dots \alpha_{k+2} \cdot \alpha_{k+3} \\ & \text{and} & \end{array}$$

$$\begin{array}{ccc} A\left(k\right) \leq & A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right) & < A\left(k+1\right) \\ & \text{i.e.} & \\ \alpha_1 \ldots \alpha_k \leq & \beta_n & < \alpha_1 \ldots \alpha_k \cdot \alpha_{k+1} \end{array}$$

Thus;

$$\begin{array}{rcl} 1 < & \frac{A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+2\right)}{\beta_n} & < \dfrac{\alpha_1 \ldots \alpha_{k+3}}{\alpha_1 \ldots \alpha_k} \\ \text{i.e. } 1 < & \frac{A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)+2\right)}{\beta_n} & < \alpha_{k+1} \cdot \alpha_{k+2} \cdot \alpha_{k+3}. \end{array}$$

If $\limsup \alpha_n < \infty$ then the sequences $A\left(S^{-1}\left(\frac{\ln n}{\ln N}+2\right)\right)$ and (β_n) are equivalent. Thus $\exists$ M s.t. $\gamma_n.e^{-M.\beta_n} \to 0$ as $n \to \infty$ so;

$$(\gamma_n) \in \Gamma\left(\Lambda_\infty\left(\beta\right)\right) = \Lambda_\infty\left(\beta\right)'$$

Conversely assume $(\gamma_n) \in \Gamma\left(\Lambda_\infty\left(\beta\right)\right) = \Lambda_\infty\left(\beta\right)'$ where $\beta_n = A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right)$. Then $\exists$ $M > 0$ s.t. $\lim_{n\to\infty} |\gamma_n|.e^{-M.\beta_n} = 0$. Show that $(\gamma_n) \in \Gamma\left(\varepsilon\left(K\right)\right)$, i.e. $\forall p$ $\exists q$ s.t. $\forall C$ $\beta\left(U_p, U_q, C\gamma_n\right) \leq n$, for n large enough. Given n, let $k = k(n)$ be such that $k \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) < k+1$.
Since $\limsup_{n\to\infty} \alpha_n < \alpha < N < \infty$. Since $\lim_{q\to\infty} \frac{q}{(q+1)^{\frac{\ln \alpha}{\ln N}}} = +\infty$, so we have an integer q_0 s.t. $\forall q \geq q_0$

$$2M\alpha^4 \leq \frac{q}{(q+1)^{\frac{\ln \alpha}{\ln N}}}. \tag{3.23}$$

Given p, choose q such that $q \geq 2p$ and $q \geq q_0$. Then $q/2 \leq q-p$ and (3.23) holds. Since $(q+1)^{\frac{\ln \alpha}{\ln N}} = \alpha^{\frac{\ln(q+1)}{\ln N}}$;

$$(3.23) \Leftrightarrow M \cdot \alpha^{4+\frac{\ln(q+1)}{\ln N}} < q/2 \tag{3.24}$$

Choose an integer μ such that

$$N^\mu \leq q+1 < N^{\mu+1}.$$

Then $\mu \leq \frac{\ln(q+1)}{\ln N}$. So (3.23) $\Rightarrow$

$$M.\alpha^{4+\mu} \leq M.\alpha^{4+\frac{\ln(q+1)}{\ln N}} \leq q/2 \leq q-p \Rightarrow \tag{3.25}$$

$$\begin{aligned} M \cdot \alpha_{k-\mu-2} \ldots \alpha_{k+1} &\leq q-p \\ \Rightarrow M \cdot \alpha_1 \alpha_{k+1} &\leq (q-p) \cdot \alpha_1 \ldots \alpha_{k-\mu-3} \\ \Rightarrow M \cdot \alpha_1 \ldots \alpha_{k+1} &\leq (q-p) \cdot \alpha_1 \ldots \alpha_{k-\mu-3} \\ \text{or } M \cdot A(k+1) &\leq (q-p) \cdot A(k-\mu-3) \end{aligned}$$

since $k-\mu-3 = k-(\mu+1)-2 \leq S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} - 2 \Rightarrow$

$$M.A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)\right) \leq M.A(k+1) \leq (q-p).A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} - 2\right) \tag{3.26}$$

Given $C>0$, for large n, $C|\gamma_n| \leq e^{M\beta_n}$. So

$$\gamma_n \leq e^{M.\beta_n} \leq e^{(q-p)A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right)-\frac{\ln(q+1)}{\ln N}-2\right)} \tag{3.27}$$

which implies that

$$\begin{aligned} \left(\frac{\ln C|\gamma_n|}{q-p}\right) &\leq A\left(S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} - 2\right) \\ \Rightarrow A^{-1}\left(\left(\frac{|\ln C\gamma_n|}{q-p}\right)\right) &\leq S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} - 2 \\ \Rightarrow A^{-1}\left(\left(\frac{\ln|C\gamma_n|}{q-p}\right)\right) + 2 &\leq S^{-1}\left(\frac{\ln n}{\ln N}\right) - \frac{\ln(q+1)}{\ln N} \\ \Rightarrow N^{A^{-1}\left(\left(\frac{\ln|C\gamma_n|}{q-p}\right)\right)+2} &= e^{\left(A^{-1}\left(\ln\left(\frac{|C\gamma_n|}{q-p}\right)\right)+2\right).\ln N} \leq \\ N^{S^{-1}\left(\frac{\ln n}{\ln N}\right)-\frac{\ln(q+1)}{\ln N}} &= \frac{N^{S^{-1}\left(\frac{\ln n}{\ln N}\right)}}{(q+1)} \\ \Rightarrow (q+1).N^{A^{-1}\left(\frac{\ln\gamma_n}{q-p}\right)+2} &\leq N^{S^{-1}\left(\frac{\ln n}{\ln N}\right)} \leq \\ N^{\frac{\ln n}{\ln N}} &= n. \end{aligned} \tag{3.28}$$

The last inequality equality follows from the fact that $S^{-1}(x) \leq x$ for large x. This can be seen as follows; $S(k+1) - S(k) = T_k \nearrow \infty$, so $S(k+1) \geq S(k) + 1$ for large k.
So by (3.22) we get $\beta\left(U_p, U_q, |\gamma_n|\right) \leq n$ for large n, which is equivalent to saying that $(\gamma_n) \in \Gamma\left(\varepsilon\left(K\right)\right)$ i.e.,

$$\Gamma\left(\varepsilon\left(K\right)\right) = \Gamma\left(\Lambda_\infty\left(\beta\right)\right).$$

Bibliography

[1] B. Arslan, A. P. Goncharov and M. Kocatepe, “Spaces of Whitney Functions on Cantor-Type Sets”, (Canadian J. Math. 54(2002), 225-238).

[2] A. Aytuna, J. Krone and T. Terzioğlu, “Complemented Infinite Type Power Series Spaces of Nuclear Fréchet Spaces”, (Math. Ann. 283(1989), 193-202).

[3] C. Bessaga, A. Pelczyński and S. Rolewicz, “On Diametral Approximative Dimension and Linear Homogeneity of F-spaces”, (Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9(1961), 677-683).

[4] E. Dubinsky, “Basic sequences in (s)”, (Studia Math. 59 (1976/77), no. 3, 283-293).

[5] A. Goncharov, “Spaces of Whitney Functions with Basis”, (Math. Nachr. 220 (2000), 45-57).

[6] A. Goncharov and M. Kocatepe, “ Isomorphic Classification of Spaces of Whitney Functions”, (Michigan Math. J. 44(1997), 555-577).

[7] B. Malgrange, “Ideals of Differentiable Functions”, (Oxford Univ. Press, London, 1967).

[8] R. Meise and D. Vogt, “Introduction to Functional Analysis”, (Clarendon Press, Oxford, 1997).

[9] B. S. Mityagin, “Approximative Dimension and Bases in Nuclear Spaces”, (Russian Math. Surveys (4) 16(1961), 59-127).

[10] B. S. Mitjagin, N. M. Zobin, “Contre-example à l’existence d’une base dans un espace de Fréchet nucléaire” ((French) C. R. Acad. Sci. Paris Sér. A 279 (1974), 325-327)

[11] A. Pietsch, “Nuclear Locally Convex Spaces”, (Springer Verlag, 1972).

[12] A.P. Robertson and W. Robertson, “Topological Vector Spaces”, (Cambridge University Press, 1973).

[13] T. Terzioğlu, “On the Diametral Dimension of Some Classes of F-spaces”, (J. Karadeniz Univ. Fac. Arts Sci. Ser. Math.-Phys. 8 (1985), 1-13).

[14] T. Terzioğlu, “Stability of smooth sequence spaces”, (J. reine angew. Math. 276(1975), 184-189).

[15] M. Tidten, “Fortsetzungen von $C^\infty-$Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}$ definiert sind”, (Manuscripte Math. 27(1979), 291-312).

[16] V. M. Tikhomirov, “On n-th diameters of compact sets”, (Dokl. Akad. nauk SSSR (4) 130(1960), 734-737).

[17] D. Vogt, “Charakterisierung der Unterräume von s”, (Math. Z. 155(1977), 109-117).

[18] D. Vogt and M. J. Wagner, “Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau”, (Studia Math. 67 (1980), 225-240).

[19] Y. C. Wong, “Schwartz Spaces, Nuclear Spaces and Tensor Products”, (Springer-Verlag, 1979)

[20] V. Zahariuta, “Linear topologic invariants and their applications to isomorphic classification of generalized power spaces”, (Turkish J. Math. 20, no. 2, 237-289(1996)).

[21] V. P. Zahariuta, “Linear Topological Invariants and the Isomorphism of Spaces of Analytic Functions”, (Matem. analiz i ego pril./ Rostov-on-Don, Rostov Univ., 2(1970),3-13 ; 3(1971), 176-180(Russian)).

[22] V. P. Zahariuta, “Some linear topological invariants and isomorphisms of tensor products of scale’s centers”, (Izv. Severo-kavkaz. Nauchn. Centra Vyssh. Shkoly 4(1974), 62-84).